MAXIMUM ENTROPY-BASED UNCERTAINTY MODELING AT THE FINITE ELEMENT LEVEL

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ABSTRACT

A novel approach is proposed for the modeling of uncertainties in finite element models of linear structural or thermal problems. This uncertainty is introduced at the level of each finite element by randomizing the corresponding elemental matrices (e.g., mass, stiffness, conductance) using the maximum entropy concepts. The approach is characterized by only two parameters, one expressing the overall level of uncertainty while the other is the correlation length underlying the random elemental matrices. The approach is exemplified on a structure example developed with Nastran.

INTRODUCTION

A large number of investigations have focused in the last three decades on the introduction of uncertainty in computational problems, structures in particular, see [1] for a recent review of applicable stochastic methods. Broadly speaking, one can separate the approaches into two classes: those that focus on the modeling of specific material and/or geometric properties (as random variables, processes, or fields in a probabilistic context), and those that proceed directly at the level of a reduced order model (ROM) of the finite element discretization. These latter methods benefit in particular from the computational efficiency of ROMs and thus are quite attractive. One challenge they bring out is the modeling of the ROM parameters, often regrouped in matrices, e.g., mass and stiffness matrices of linear structural problems.

This modeling issue was elegantly resolved by Soize [2] who proposed to apply a maximum entropy concept to the random matrices, that is the joint probability density function of the random matrix elements should be selected to (i) satisfy physical requirements (such as positive definiteness, symmetry, non singularity), (ii) ensure that the mean of the matrices corresponds to the matrix of the baseline deterministic model, and (iii) maximize the entropy given the constraints of (i) and (ii). This broad concept has been applied to matrices and fields with a variety of properties [1-14] for which the construction of realizations has been detailed and is often very straightforward permitting a broad set of applications, see [3] for review.

A key aspect of the maximum entropy approach is that it does not correspond to uncertainty on specific parameters (thickness, Young's modulus, etc.) of the computational model but rather on the global properties of the reduced order model. Accordingly, this approach does capture some epistemic uncertainty, in addition to the aleatoric one. To exemplify this aspect, consider the modeling of a plate the mean model of which is flat and symmetric through the thickness. Adopting as ROM basis vectors a series of linear modes of the transverse and in plane motions leads to a mean ROM stiffness matrix which exhibits a series of zeros representative of the decoupling of the in plane and transverse motion. The application of the maximum entropy approach (e.g., according to [2]) leads to random matrices which are fully populated, i.e., corresponding to structures with in plane – transverse coupling. That is, the

randomization has generated models of plates exhibiting asymmetry through thickness and/or curvature at the contrary of the mean model but likely consistently with a manufactured version of this mean model.

While the strategy to impose particular properties on the random ROM *matrices* is clear, i.e., redo the calculus of variation effort involved in the optimization of the entropy, it is much more difficult to impose particular features to the ROM *solution*. Such an issue has been encountered recently [14] in connection with structures exhibiting a localized structural or thermal response. More specifically, it was expected on physical grounds that the response would remain localized to a fraction of the structure, however, the application of the maximum entropy to ROMs lead to a globalization of a response as can be expected from a high entropy solution.

A modification of the maximum entropy modeling approach was developed for the above cases by understanding the properties of the ROM matrices leading to a localization of the response and adapting the method to meet those properties. Another approach, however, would be to introduce the uncertainty in a more spatially localized manner to mitigate the globalization associated with the maximization of the entropy. It is such an approach which is proposed here, i.e., to introduce the uncertainty on each finite element matrix (mass, stiffness, conduction, etc.) following the maximum entropy concept and treating the corresponding matrix from the baseline model as a mean ROM. This approach represents a novel compromise between the modeling of uncertainty within the elements (e.g., by randomizing the elasticity tensor) on one end and at the level of a global ROM on the other. It brings some epistemic uncertainty not present in the former approach while retaining more local character than the latter one.

Validations of this approach to a structural problem is carried out below. It is moreover demonstrated that the approach can be used to model the uncertainty in the elasticity tensor to relate element/nodal strains to their stress counterparts.

MAXIMUM ENTROPY APPROACH FOR POSITIVE DEFINITE/SYMMETRIC MATRICES

The original formulation of the maximum entropy approach focuses on symmetric positive definite matrices $\boldsymbol{A}$ for which it assumes that the mean $\overline{\boldsymbol{A}}$ is known. This limited information is not sufficient to uniquely define the joint probability density function of the elements of $\boldsymbol{A}$, denoted as $p_{\boldsymbol{A}}(\boldsymbol{a})$. Faced with this issue, Soize proposed in [2] that this joint probability density function be selected as the one that maximizes entropy S given the available information. That is, $p_{\boldsymbol{A}}(\boldsymbol{a})$ should maximize

$$S = -\int_{\Omega} p_{\boldsymbol{A}}(\boldsymbol{a}) \ln\, p_{\boldsymbol{A}}(\boldsymbol{a}) d\boldsymbol{a} \tag{1}$$

given that

unit total probability:
$$\int_{\Omega} p_{\boldsymbol{A}}(\boldsymbol{a})\, d\boldsymbol{a} = 1 \tag{2}$$

given mean:
$$\int_{\Omega} \boldsymbol{a}\; p_{\boldsymbol{A}}(\boldsymbol{a})\, d\boldsymbol{a} = \overline{\boldsymbol{A}} \tag{3}$$

nonsingularity: $$\int_{\Omega} \ln[\det(\boldsymbol{a})] p_{\boldsymbol{A}}(\boldsymbol{a})\, d\boldsymbol{a} = \nu \text{ finite} \tag{4}$$

where the domain of support Ω of the obtained probability density function is then such that the matrix $\boldsymbol{A}$ is positive definite, or equivalently that it admits a Cholesky decomposition, i.e.,

$$\Omega = \left\{ \boldsymbol{a} = \tilde{\boldsymbol{L}}\tilde{\boldsymbol{L}}^T ; \tilde{L}_{ij}, i, j = 1,...,n : \left[\tilde{L}_{ij} \in (-\infty,+\infty), i > j\right] \cap \left[\tilde{L}_{ii} \in [0,+\infty)\right] \right\}. \tag{5}$$

The probability density function $p_{\boldsymbol{A}}(\boldsymbol{a})$ maximizing *S* given the constraints of Eqs (2)-(4) can be derived by calculus of variation and is found to be

$$p_{\boldsymbol{A}}(\boldsymbol{a}) = \tilde{C}[\det(\boldsymbol{a})]^{\lambda-1} \exp\left[-\operatorname{tr}\left(\tilde{\mu}^T \boldsymbol{a}\right)\right] \tag{6}$$

where $\tilde{C}$ is the appropriate normalization constant to satisfy the normalization condition, Eq. (2) and μ and λ are the Lagrange multipliers associated with the constraints of Eqs (3) and (4), respectively. After a change of random variables, it is found that the matrices *A* of joint probability density function $p_{\boldsymbol{A}}(\boldsymbol{a})$, Eq. (6), can be generated as

$$\boldsymbol{A} = \bar{\boldsymbol{L}}\, \boldsymbol{H}\, \boldsymbol{H}^T\, \bar{\boldsymbol{L}}^T \tag{7}$$

where $\bar{\boldsymbol{L}}$ is any decomposition, e.g., Cholesky, of $\bar{\boldsymbol{A}}$, i.e.,

$$\bar{\boldsymbol{A}} = \bar{\boldsymbol{L}}\, \bar{\boldsymbol{L}}^T \tag{8}$$

Moreover, $\boldsymbol{H}$ is a lower triangular matrix such that (see also Fig. 1)

(1) its off-diagonal elements H_{il}, $i \neq l$, are normally distributed (Gaussian) random variables with standard deviation $\sigma = 1/\sqrt{2\mu}$, and

(2) its diagonal elements H_{ii} are obtained as $H_{ii} = \sqrt{Y_{ii}/\mu}$ where Y_{ii} is Gamma distributed with parameter $(p(i)-1)/2$ where

$$p(i) = n - i + 2\lambda_0 - 1 \text{ and } \quad \mu = (n + 2\lambda_0 - 1)/2 \tag{9}$$

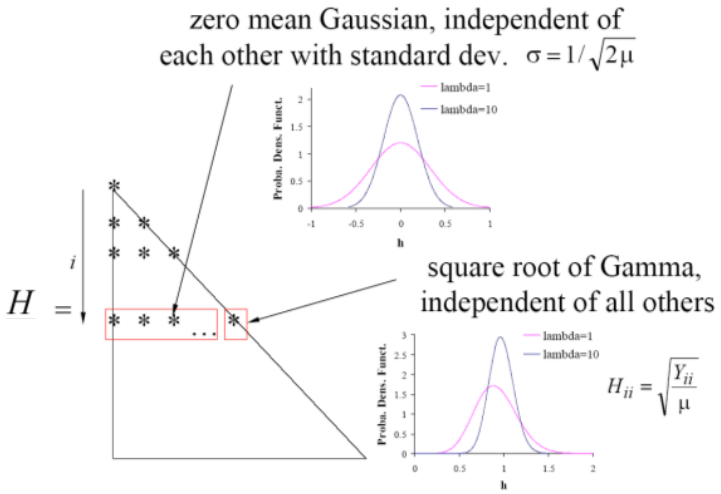


Figure 1. Structure of the random $\boldsymbol{H}$ matrices with n = 8, i =2, and λ_0=1 and 10.

In the above equations, *n* is the size of the matrices and the parameter $\lambda_0 > 0$ is the free parameter of the statistical distribution of the random matrices $\boldsymbol{A}$. An alternative

parametrization is through the dispersion parameter δ defined as

$$\delta^2 = \frac{n+1}{n+2\lambda_0 - 1}. \tag{10}$$

ELEMENTAL LEVEAL UNCERTAINTY MODELING

As described in the introduction, the focus of the present effort is on developing an uncertainty modeling strategy at the level of the finite element. To this end, denote by $\overline{\boldsymbol{K}}$ an elemental matrix (stiffness, mass, conductance) of the baseline finite element. Next, assume that the only properties required for its uncertain counterpart $\boldsymbol{K}$ are that this matrix is positive definite, symmetric, and non-singular. Then, following the maximum entropy strategy, one can express

$$\boldsymbol{K} = \overline{\boldsymbol{L}}_{\boldsymbol{K}}\, \boldsymbol{H}_{\boldsymbol{K}}\, \boldsymbol{H}_{\boldsymbol{K}}^T\, \overline{\boldsymbol{L}}_{\boldsymbol{K}}^T \tag{11}$$

where $\overline{\boldsymbol{L}}_{\boldsymbol{K}}$ is a decomposition of $\overline{\boldsymbol{K}}$ satisfying

$$\overline{\boldsymbol{K}} = \overline{\boldsymbol{L}}_{\boldsymbol{K}}\, \overline{\boldsymbol{L}}_{\boldsymbol{K}}^T \tag{12}$$

and $\boldsymbol{H}_{\boldsymbol{K}}$ is a lower triangular matrix as defined in Fig. 1. The process could then be repeated for each element in turn.

In applying the above concepts, there are two key issues which must be carefully addressed. The first one is that *the matrices* $\boldsymbol{H}_{\boldsymbol{K}}$ *corresponding to different elements cannot be simulated independently of each other*. Doing so would induce very high spatial frequency variations which are unphysical. Rather, it is proposed here to adopt the matrix field modeling proposed in [7] which views each element H_{ij} as the transformation of a zero mean, unit variance Gaussian field P_{ij} with a specified stationary autocorrelation function

$$R(\boldsymbol{y}) = E\left[P_{ij}(\boldsymbol{x})P_{ij}(\boldsymbol{x}')\right] \qquad \boldsymbol{y} = \boldsymbol{x}' - \boldsymbol{x} \tag{13}$$

where $\boldsymbol{x}$ and $\boldsymbol{x}'$ denote the coordinates of two elements (e.g., of their center).

Specifically, for $i \neq j$

$$H_{ij} = \sigma P_{ij} \tag{14}$$

while for $i = j$

$$H_{ii} = F_{H_{ii}}^{-1}\left[F\left(P_{ij}\right)\right] \tag{15}$$

where F is the cumulative distribution function of the standard Gaussian random variable and $F_{H_{ii}}^{-1}$ is the inverse of the cumulative distribution function of the Gamma random variable H_{ii}, see previous section.

The correlation between random elemental matrices of neighboring finite elements implied by the above algorithm must be reflected on every component of the *assembled* matrix. That is, if a strong correlation is expected between two different finite elements, then there must exist a similarly strong correlation between the components of their elemental matrices *which are added together in the construction of the global matrix*. Since the matrix $\boldsymbol{H}_{\boldsymbol{K}}$ is built from the independent fields P_{ij}, this condition can be satisfied if:

(1) the elemental matrices of the mean and uncertain models are expressed in the *same* (i.e., global) frame of reference, and

(2) each simulated sample of the random global matrix is independent of the ordering of the

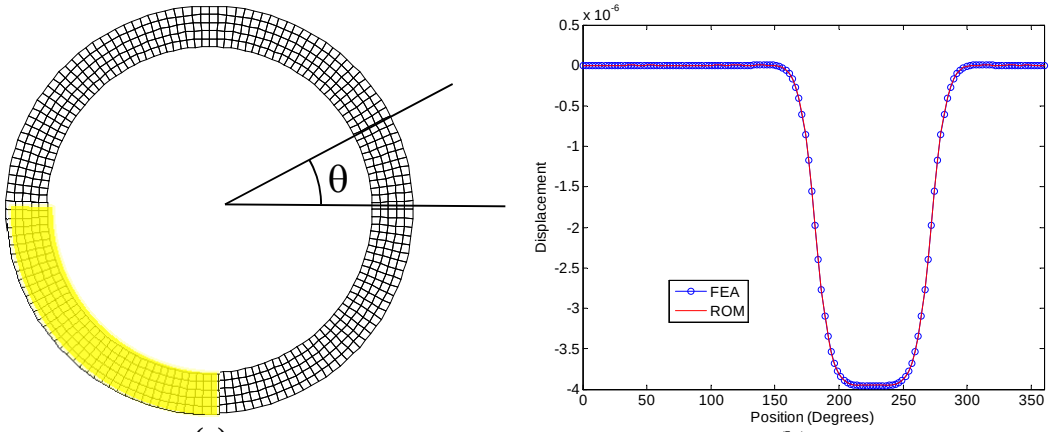
nodes in each element.
This latter condition can be achieved as follows.
(i) Organize the mean elemental matrices $\overline{\boldsymbol{K}}$ consistently with the ordering of its degrees of freedom as: degree of freedom 1 for all nodes, degree of freedom 2 for all nodes, etc.
(ii) Adopt lower triangular (or block lower triangular) decompositions $\overline{\boldsymbol{L}}_{\boldsymbol{K}}$ to retain the same ordering of the degrees of freedom.
(iii) Restrict the random matrices $\boldsymbol{H}_{\boldsymbol{K}}$ so that each of their samples is invariant with respect to a permutation of the ordering of the nodes. This is achieved here by expressing

$$\boldsymbol{H}_{\boldsymbol{K}} = \boldsymbol{H} \otimes \boldsymbol{I}_{\boldsymbol{r}} \tag{16}$$

where $\boldsymbol{H}$ is a *m*x*m* random matrix simulated as described in Eq. (13)-(15) where *m* is the number of degrees of freedom per node. Moreover, $\boldsymbol{I}_{\boldsymbol{r}}$ denotes the *r*x*r* identity matrix where *r* is the number of nodes per element and $\otimes$ denotes the Kronecker product operation.
Once the elemental matrix (or matrices) have been simulated for each element, the finite element model is reassembled and the response can be determined. Proceeding with a series of such simulations provides a population of responses from which statistics can be determined.

APPLICATION EXAMPLE

To illustrate the above uncertainty modeling, consider the annulus shown in Fig. 2(a) of inner radius 0.8m, outer radius 1m, thickness 0.002m clamped on its inner radius and free on the outer one. The material, aluminum, is assumed to be homogenous and isotropic with Young's modulus E = 7.3 10^{10} Pa and Poisson's ratio ν= 0.316. The annulus is subjected to a static uniform unit pressure in the quadrant $\theta \in [180,270]$ degrees highlighted in yellow in Fig. 2(a). To evaluate the displacement field of the annulus, it was modeled by 4-node shell finite elements within Nastran (CQUAD4 elements) with a mesh of 144 nodes around the periphery and 6 in the radial direction. Then, shown in Fig. 2(b) is the transverse displacement of the periphery which is clearly localized near the excitation, i.e., in the band $\theta \in [150,300]$ degrees.



(a) (b)

Figure 2. (a) The annulus and its finite element model with the loading domain highlighted in yellow. (b) Static transverse displacement at the periphery, full finite element (FEA) and reduced order (ROM) models.

The autocorrelation function of [7] was selected here with a correlation length equivalent to 8

elements and a value $\delta = 0.1$ was adopted. Then, shown in yellow in Fig. 3(a) is the uncertainty band corresponding to the 5th and 95th percentile of the transverse displacement of the periphery as determined from the 300 samples of the response. The response is clearly localized as the one from the mean model. For comparison, shown in Fig. 3(b) is the uncertainty band generated by the approach of [14]. While a one to one comparison of the bands cannot be made as the methods involve different parameters, it is clear that the predictions are similar in all qualitative aspects.

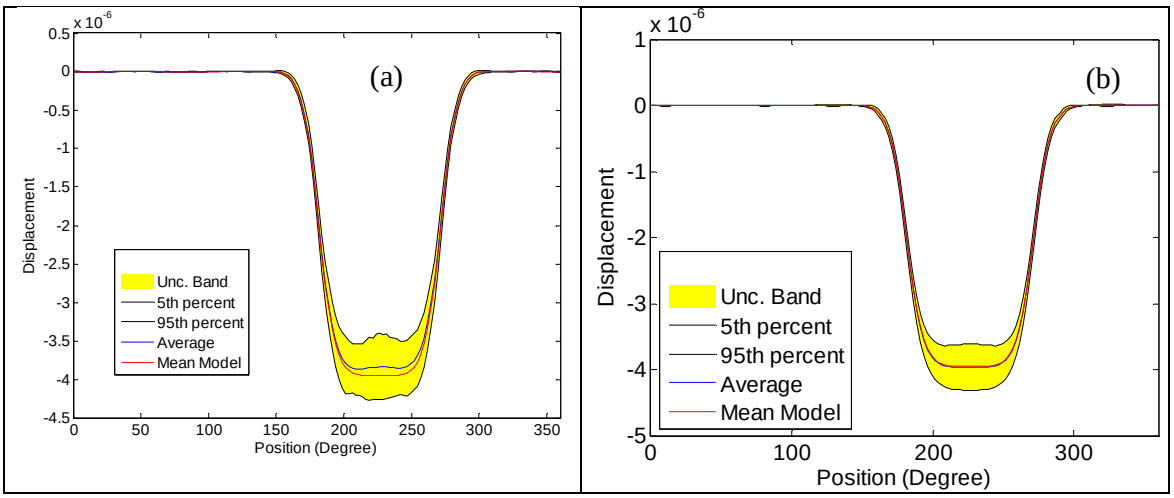


Figure 3. Static transverse displacement at the periphery of the mean annulus (in red) and 5th-95th percentile uncertainty band from (a) the above uncertain finite element model, (b) [14].

The application of the above elemental uncertainty modeling approach to the strain-stress transformation will be discussed in the final version of the paper.

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